International Journal of Theoretical Physics, Vol. 34, No. 1, 1995

Effective Lagrangians in Classical Yang–Mills Theories

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Received March 16, 1994

The formalism of nonlinear electrodynamics is adapted to the Yang–Mills field theory and it is shown that vacuum polarization effects can be described classically through an effective Lagrangian. As an example, an ad hoc Lagrangian is proposed which leads to a simple solution corresponding to a linear plus Coulomb type potential.

The aim of this article is to show that the basic ideas of nonlinear electrodynamics can be adapted to Yang–Mills field theories. In particular, it is shown that by appropriately choosing an effective Lagrangian, one can obtain simple classical solutions which exhibit the confinement property. It is expected that this formalism may be useful as a simple semiclassical model of QCD interactions.

Several exact solutions of the Yang–Mills field equations are known, but they all describe classical fields which vanish asymptotically (Actor, 1979), and therefore do not reproduce one of the distinctive feature of QCD, which is the phenomena of quark confinement. This effect is due to the interaction of gluons with themselves and can be interpreted as a polarization of the vacuum. Thus, to describe confinement even at a classical level, it is necessary to include polarization effects in some effective way.

A somewhat similar problem is present in the electromagnetic theory. The interaction of photons with themselves through the exchange of an electron is well known in quantum field theory. At a semiclassical level, it can be interpreted as a nonlinear electromagnetic effect related to the polarization of the vacuum. In a classical work, Schwinger (1951) obtained the effective Lagrangian which includes such polarization effects.

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0020-7748/95/0100-0047$07.50/0

Nonlinear electrodynamics was developed many years ago as a model which represents quantum effects within an essentially classical formalism. The basic work on this subject is due to Born and Infeld (1934), but the formalism has been developed since then by many authors [see, e.g., Peres (1961); for a detailed exposition, see Plebanski (1968)].

The following formalism is a straightforward generalization from nonlinear electrodynamics to Yang–Mills theory. Given a gauge symmetry and potentials A^a_μ, one defines the field

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu \tag{1}$$

where f_{abc} are the structure constants of the gauge group. Now, the basic idea is that the Lagrangian $\mathscr{L}$ of the field can be in principle a general function of the Lorentz- and gauge-invariant scalars of the field. There are two such scalars, which can be written in the convenient form

$$\begin{aligned} F^a_{\mu\nu} F^{\mu\nu}_a &\equiv 2(B^2 - E^2) \\ F^{*a}_{\mu\nu} F^{\mu\nu}_a &\equiv 4EB \end{aligned} \tag{2}$$

Given a Lagrangian $\mathscr{L}(E, B)$, the next step is to define the polarization tensor as

$$P^a_{\mu\nu} = -2 \frac{\partial \mathscr{L}}{\partial F^{\mu\nu}_a} \tag{3}$$

or, more explicitly,

$$P^a_{\mu\nu} = \frac{1}{E^2 + B^2} \left[F^a_{\mu\nu} \left(E \frac{\partial}{\partial E} - B \frac{\partial}{\partial B} \right) - F^{*a}_{\mu\nu} \left(B \frac{\partial}{\partial E} + E \frac{\partial}{\partial B} \right) \right] \mathscr{L} \tag{4}$$

The variation of the Lagrangian, including interaction with a current J^μ_a, gives the equation

$$\nabla_\nu P^{\mu\nu}_a + g f_{\alpha bc} A^b_\nu P^{\mu\nu}_c = 4\pi g J^\mu_a \tag{5}$$

Finally, the energy-momentum tensor of the field turns out to be

$$T_{\mu\nu} = \frac{1}{4\pi} [F^a_{\mu\alpha} P^{a}{}_{\nu}{}^{\alpha} + g_{\mu\nu} \mathscr{L}] \tag{6}$$

Summing up, given a Lagrangian $\mathscr{L}(E, B)$, equations (1), (4), and (5) provide a complete set of field equations. An explicit example will be worked out in the following.

In the case of a point like color charge, the current can be taken as $J^\mu_a = Q\, \delta^1_a \delta(r)$. Then, the simplest ansatz has the form

$$A^\mu_1 = (\phi, 0) \qquad \text{and} \qquad A^\mu_i = 0 \qquad \text{if} \quad i \neq 1 \tag{7}$$

A solution corresponding to this ansatz in the case of a standard Lagrangian was obtained by Ikeda and Miyachi (1962): it is essentially a static Coulomb-type potential. Consider, however, an ad hoc Lagrangian of the form

$$\mathscr{L} = \tfrac{1}{2}(E^2 - B^2) - k(E^2 - B^2)^{1/2} \tag{8}$$

where k is a constant. The ansatz (7) implies that the only nonzero component of the field $F^a_{\mu\nu}$ (in spherical coordinates) is $F^1_{tr} = E$. On the other hand, equation (5) implies that

$$\frac{1}{r^2}\frac{d}{dr}(r^2 P^{tr}_1) = 4\pi g Q\,\delta(r) \tag{9}$$

from which it follows that

$$P^{tr}_1 = -\frac{d}{dr}\left(\frac{gQ}{r}\right) \tag{10}$$

The solution (for $r \neq 0$) has the form

$$\phi = -\frac{Q}{r} + kr \tag{11}$$

and $E = (Q/r^2) + k$. Thus the Lagrangian (6) leads to a Cornell-type potential for the color field (Eichten *et al.*, 1980).

The energy-momentum tensor can be easily calculated. The only nonvanishing terms are diagonal:

$$T_{tt} = -T_{rr} = \frac{1}{8\pi}\left(\frac{Q}{r^2} + k\right)^2 \tag{12}$$

$$T^\theta{}_\theta = T^\phi{}_\phi = \frac{1}{8\pi}\left(\frac{Q^2}{r^4} - k^2\right) \tag{13}$$

Note that the energy density is positive definite and the tangential pressures become tensions for $r > (Q/k)^{1/2}$.

The above solution is not truly a Yang–Mills solution, since the non-Abelian terms in the field equations have played no role. Actually, the linearly increasing term in the potential is entirely due to the form of the effective Lagrangian. However, the formalism should permit one to obtain more general and interesting solutions.

An important problem which remains open is to find the form of the effective Lagrangian from general principles, instead of proposing an ad hoc one. Once an appropriate Lagrangian is found, the present formalism can be applied quite straightforwardly.

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